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On CLT and non-CLT groups

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Abstract
In this note, we prove that for every integer $d \geq 2$ which is not a prime power, there exists a finite solvable group G such that $d \mid |G|$, $\pi(G) = \pi(d)$ and G has no subgroup of order d. We also introduce the CLT-degree of a finite group and answer two questions about it.

Keywords CLT group · Solvable group · Frobenius group

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1 Introduction

All groups considered in this note are finite. A group is said to be CLT if it possesses subgroups of every possible order (that is, it satisfies the Converse of Lagrange's Theorem) and non-CLT otherwise. It is well-known that CLT groups are solvable (see [11]) and that supersolvable groups are CLT (see [10]). Moreover, the inclusion between the classes of CLT groups and solvable groups, as well as the inclusion between the classes of supersolvable groups and CLT groups are proper (see, for example, [3]). Recall also the papers [1, 2, 13] which construct non-CLT groups of order $p^{\alpha}q^{\beta}$ with p, q primes and $\alpha, \beta \in \mathbb{N}^*$.

The first starting point for our work is given by the smallest examples of non-CLT groups, namely A_4 and $\mathrm{SL}(2,3)$. It is well-known that they have no subgroup of order 6 and 12, respectively. This leads to the following natural question:

$$\textit{Given an integer } d \geq 2, \textit{ is it possible to construct a group}$$
$$\textit{whose order is divisible by } d, \textit{ but having no subgroup of order } d?$$

Obviously, the answer to this question is "NO" if d is a prime power. So, in what follows we will assume that d is a composite number and we will denote by $\pi(d)$ the set of primes dividing d. Also, for a finite group G we will denote $\pi(G) = \pi(|G|)$.

Our first result shows that the answer to the above question is "YES" for all composite numbers d.

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Theorem 1.1 *For every integer $d \geq 2$ which is not a prime power, there exists a finite solvable group G such that $d \mid |G|$, $\pi(G) = \pi(d)$ and G has no subgroup of order d.*

The second starting point for our work is given by a question of Martino Garonzi on MathOverflow [4]:

$$\textit{Is there a constant } c > 0 \textit{ such that } \frac{D(G)}{\tau(|G|)} > c \textit{ for any finite group } G?$$

Here $D(G)$ denotes the number of divisors d of $|G|$ for which there exists a subgroup of G of order d and $\tau(|G|)$ denotes the number of all divisors of $|G|$. Note that this question remained unanswered for more than ten years.

It suggests to consider the function

$$d_{CLT}(G) = \frac{D(G)}{\tau(|G|)},$$

which will be called the *CLT-degree* of G. Clearly, $d_{CLT}(G)$ measures the probability of a finite group G to be CLT. It is easy to see that this new function satisfies the following properties:

- $0 < d_{CLT}(G) \leq 1$, for any finite group G. Moreover, $d_{CLT}(G) = 1$ if and only if G is CLT.
- d_{CLT} is multiplicative, that is $d_{CLT}(G_1 \times G_2) = d_{CLT}(G_1)d_{CLT}(G_2)$, for any finite groups G_1, G_2 of coprime orders.
- If N is a normal subgroup of G, then

$$d_{CLT}(G/N) \leq \frac{\tau(|G|)}{\tau(|G|/|N|)}\, d_{CLT}(G).$$

- If $|G| = p_1^{n_1} \cdots p_k^{n_k}$ with $k \geq 2$, then

$$d_{CLT}(G) \geq \frac{\sum_{i=1}^k n_i + 2}{\prod_{i=1}^k (n_i + 1)}.$$

  Moreover, we have equality if and only if G has all proper subgroups of prime power order. Such a group is either cyclic of order pq, where p and q are distinct primes, or a semidirect product $C_p^n \rtimes C_q$, where C_q acts irreducibly on C_p^n. Thus

$$d_{CLT}(C_p^n \rtimes C_q) = \frac{n+3}{2n+2}$$

  and, in particular, $d_{CLT}(A_4) = \frac{5}{6}$.

Let $\mathcal{G}$ be the class of all finite groups. Our second result is stated as follows.

Theorem 1.2 *The set*

$$\mathrm{Im}(d_{CLT}) = \{d_{CLT}(G) \mid G \in \mathcal{G}\}$$

is dense in $[0, 1]$.

Obviously, Theorem 1.2 gives a negative answer to Garonzi's question.

Corollary 1.3 *There is no constant $c > 0$ such that $\frac{D(G)}{\tau(|G|)} > c$ for any finite group G.*

We also infer that:

Corollary 1.4 *There is no constant $c < 1$ such that $d_{CLT}(G) \leq c$ for any finite non-CLT group G.*

Note that an example of a family of finite non-CLT groups $(G_n)_{n\geq 1}$ with $\lim_{n\to\infty} d_{CLT}(G_n) = 1$ will be given at the end of Sect. 3.

Finally, we formulate a natural open problem related to our results.

Open problem Is it true that $\lim_{n\to\infty} d_{CLT}(S_n) = 0$?

Note that we have

$$d_{CLT}(S_n) \leq \frac{\#iso(S_n)}{\tau(n!)} \leq \frac{\#ccs(S_n)}{\tau(n!)},$$

where $\#iso(S_n)$ and $\#ccs(S_n)$ denote the number of isomorphism/conjugacy classes of subgroups of S_n, but we were not able to decide whether these ratios tend to 0 when n tends to infinity.

Most of our notation is standard and will usually not be repeated here. For basic notions and results on groups we refer the reader to [6, 7].

2 Proof of Theorem 1.1

First of all, we present two auxiliary results. Recall that a *Frobenius group NH* with kernel N and complement H can be characterized as a finite group that is a semidirect product of a normal subgroup N by a subgroup H such that $C_N(h) = 1$ for every $h \in H \setminus \{1\}$. An important example of such a group is the group $\mathrm{AGL}(1, q) = \mathbb{F}_q \rtimes \mathbb{F}_q^{\times}$ of affine linear transformations of the finite field $\mathbb{F}_q$.

Lemma 2.1 ([5]) *Let G be a Frobenius group with kernel N and K be a subgroup of G. Then one of the following holds:*

(1) $K \subseteq N$.
(2) $K \cap N = 1$.
(3) *K is a Frobenius group with kernel $N \cap K$.*

Lemma 2.2 [8] *Let G be a Frobenius group with kernel N and complement H. Then $|H| \mid |N| - 1$.*

We are now able to prove our first main result.

Proof of Theorem 1.1. We will proceed by induction on d. For $d = 6$ we can choose $G = A_4$. Now, let $d \geq 6$ and assume the statement to be true for every $d' < d$. Let $d = p_1^{n_1} \cdots p_k^{n_k}$ be the decomposition of d as a product of distinct prime factors, where $k \geq 2$. We distinguish the following two cases.

Case 1 $k = 2$

Then $d = p_1^{n_1} p_2^{n_2}$. Let $a = \exp_{p_2^{n_2}}(p_1)$ and $b = \exp_{p_1^{n_1}}(p_2)$, where $\exp_{p_i^{n_i}}(p_j)$ is the multiplicative order of p_j modulo $p_i^{n_i}$. Then either $a \nmid n_1$ or $b \nmid n_2$. Indeed, if $a \mid n_1$ and $b \mid n_2$, then $p_2^{n_2} \mid p_1^{n_1} - 1$ and $p_1^{n_1} \mid p_2^{n_2} - 1$, implying that $p_2^{n_2} < p_1^{n_1}$ and $p_1^{n_1} < p_2^{n_2}$, a contradiction.

Assume that $a \nmid n_1$ and let r be a positive integer such that $(r-1)a < n_1 < ra$. Then the Frobenius group

$$\mathrm{AGL}(1, p_1^{ra}) = C_{p_1}^{ra} \rtimes C_{p_1^{ra}-1}$$

has no subgroup of order d. Indeed, if there is $K \leq \mathrm{AGL}(1, p_1^{ra})$ with $|K| = d$, then K must be a Frobenius group with kernel of order $p_1^{n_1}$ and complement of order $p_2^{n_2}$ by Lemma 2.1.

Then Lemma 2.2 leads to $p_2^{n_2} \mid p_1^{n_1} - 1$ and thus $a \mid n_1$, a contradiction. Now, it is clear that $\mathrm{AGL}(1, p_1^{ra})$ has a subgroup G of order $p_1^{ra} p_2^{n_2}$: this G is easily seen to be a group that satisfies the desired conclusions with respect to $d = p_1^{n_1} p_2^{n_2}$.

Case 2 $k \geq 3$

Let $d' = d/p_k^{n_k}$. Then d' is not a prime power and so there exists a finite solvable group G_1 such that $d' \mid |G_1|, \pi(G_1) = \pi(d')$ and G_1 has no subgroup of order d'. Let $G = G_1 \times C_{p_k^{n_k}}$. It follows that G is solvable, $d \mid |G|$ and $\pi(G) = \pi(d)$. Moreover, G has no subgroup of order d. Indeed, if $H \leq G$ has order $d = d' \cdot p_k^{n_k}$, then H possesses a subgroup H_1 of order d'. Since G is solvable, H_1 is contained in a Hall $\pi(d')$-subgroup of G, that is $H_1 \subseteq G_1$. Thus G_1 contains subgroups of order d', a contradiction.

The proof of Theorem 1.1 is now complete. □

The following example is founded on the above proof.

Example 2.3 For $d = 60 = 2^2 \cdot 3 \cdot 5$, we have $\exp_4(3) = 2 \nmid 1 = n_2$ and therefore a finite solvable group G such that $60 \mid |G|, \pi(G) = \pi(60)$ and G has no subgroup of order 60 is

$$\mathrm{SmallGroup}(360, 123) = \mathrm{AGL}(1, 9) \times C_5 = \left(C_3^2 \rtimes C_8\right) \times C_5.$$

3 Proof of Theorem 1.2

The proof of Theorem 1.2 follows the same steps as the proof of Theorem 1.1 in [9]. It is based on the next lemma which is a consequence of Proposition outlined on p. 863 of [12].

Lemma 3.1 *Let $(x_n)_{n\geq 1}$ be a sequence of positive real numbers such that $\lim_{n\to\infty} x_n = 0$ and $\sum_{n=1}^{\infty} x_n$ is divergent. Then the set containing the sums of all finite subsequences of $(x_n)_{n\geq 1}$ is dense in $[0, \infty)$.*

We will also need the following lemma.

Lemma 3.2 *Let p, q be two primes such that q is odd and $q \mid p + 1$, and let*

$$G_{p,q}^n = \left(C_p^2 \rtimes C_q\right) \times C_q^n,\ n \geq 0.$$

Then

$$d_{CLT}(G_{p,q}^n) = \frac{3n+5}{3n+6}.$$

Proof It is easy to check that $G_{p,q}^n$ has subgroups of all possible orders except $p \cdot q^{n+1}$. □

We are now able to prove our second main result.

Proof of Theorem 1.2. Let $I = \{n_1, ..., n_k\} \subset \mathbb{N}$ and let $q_1, ..., q_k$ be distinct odd primes. By Dirichlet's theorem, we can choose distinct primes $p_1, ..., p_k$ such that $q_i \mid p_i + 1$, for all $i = 1, ..., k$. We remark that these primes can be chosen such that

$$\{p_i, q_i\} \cap \{p_j, q_j\} = \emptyset, \text{ for all } i \neq j.$$

Since d_{CLT} is multiplicative, Lemma 3.2 shows that

$$d_{CLT}\left(\prod_{i=1}^{k} G_{p_i,q_i}^{n_i}\right) = \prod_{i=1}^{k} \frac{3n_i+5}{3n_i+6}$$

and so

$$A = \left\{ \prod_{n \in I} \frac{3n+5}{3n+6} \;\middle|\; I \subset \mathbb{N}, |I| < \infty \right\} \subseteq \mathrm{Im}(d_{CLT}).$$

Thus it suffices to prove that A is dense in $[0, 1]$.

Consider the sequence $(x_n)_{n\geq 1} \subset (0, \infty)$, where $x_n = \ln(\frac{3n+6}{3n+5})$. Clearly, $\lim_{n\to\infty} x_n = 0$. We have

$$\lim_{n\to\infty} \frac{x_n}{\frac{1}{n}} = \frac{1}{3}.$$

Therefore, since the series $\sum_{n\geq 1} \frac{1}{n}$ is divergent, we deduce that the series $\sum_{n\geq 1} x_n$ is also divergent. So, all hypotheses of Lemma 3.1 are satisfied, implying that

$$\overline{\left\{ \sum_{n \in I} x_n \;\middle|\; I \subset \mathbb{N}^*, |I| < \infty \right\}} = [0, \infty).$$

This means

$$\overline{\left\{ \ln\left(\prod_{n \in I} \frac{3n+6}{3n+5} \right) \;\middle|\; I \subset \mathbb{N}^*, |I| < \infty \right\}} = [0, \infty)$$

or equivalently

$$\overline{\left\{ \prod_{n \in I} \frac{3n+6}{3n+5} \;\middle|\; I \subset \mathbb{N}^*, |I| < \infty \right\}} = [1, \infty).$$

Then

$$\overline{\left\{ \prod_{n \in I} \frac{3n+5}{3n+6} \;\middle|\; I \subset \mathbb{N}^*, |I| < \infty \right\}} = [0, 1]$$

and consequently

$$\overline{A} = [0, 1].$$

The proof of Theorem 1.2 is now complete. □

Finally, we note that for fixed p, q, the above groups $(G^n_{p,q})_{n\geq 0}$ are non-CLT and satisfy $\lim_{n\to\infty} d_{CLT}(G^n_{p,q}) = 1$, providing a direct proof of Corollary 1.4.

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